Lecture Notes on Partial Differential Equations

Chapter IV

Fourier Series

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Before discussing how to solve linear partial differential equations by the method of separation of variables, it is important to summarize some of the important results on Fourier series. Joseph Fourier developed this type of series in his famous treatise on heat flow in the early 1800s.

1 Definitions

Definition 1.1. A function f is said to be *piecewise continuous* on $[a, b]$ if there exists finitely many points $a = x_1 < x_2 < \ldots < x_n = b$ such that f is continuous on $[x_i, x_{i+1}]$ and

$$\lim_{x \to x_i^+} f(x), \lim_{x \to x_{i+1}^-} f(x)$$

exist for $i = 1, 2, \ldots, n - 1$.

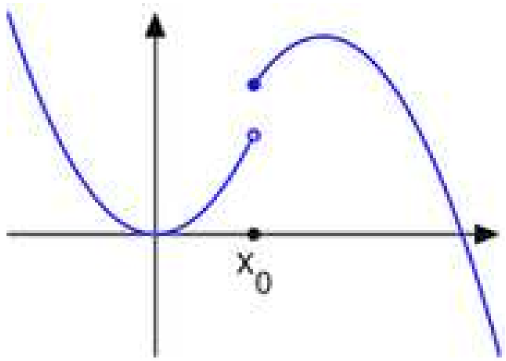

FIG. 1 – Piecewise continuous function.

Definition 1.2. A function f is said to be *periodic*, with a period p if $f(x+p) = f(x)$ for all x.

Example 1.1.
1. $f(x) = \sin(x)$ is a periodic function with period $p = 2\pi$.
2. $f(x) = \tan(x)$ is a periodic function with a period $p = \pi$.

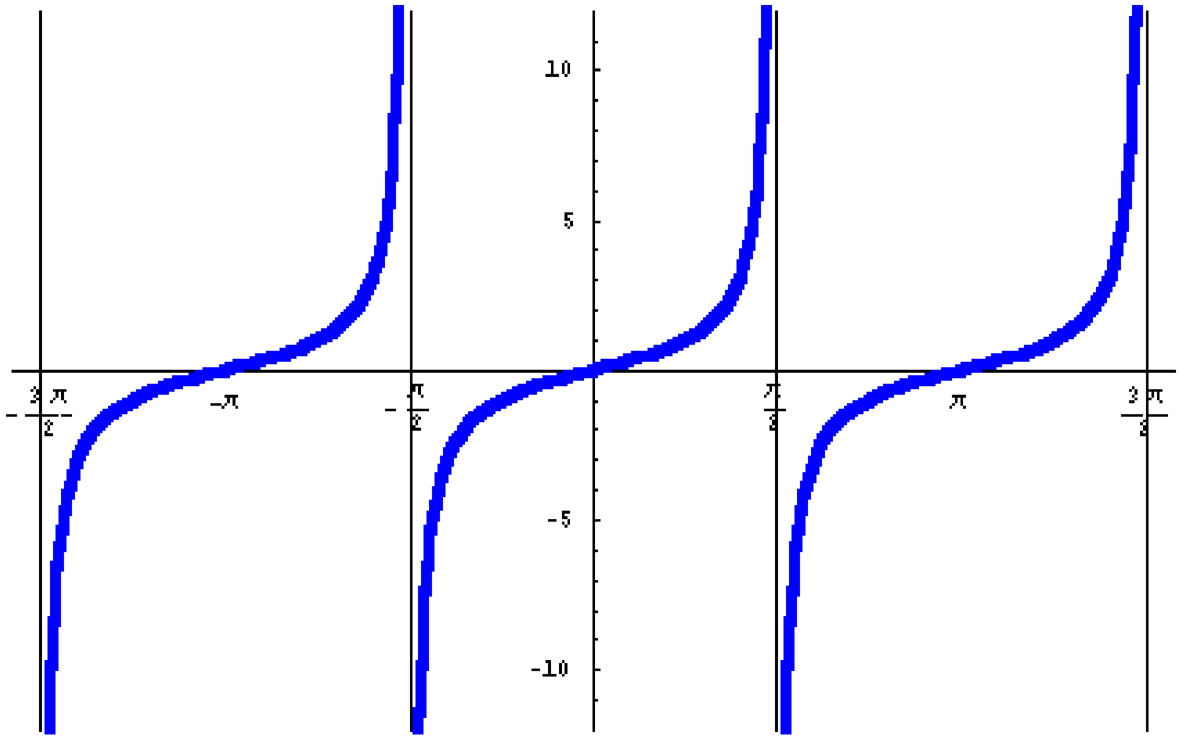

FIG. 2 – Periodic function.

Definition 1.3. Two functions f and g are said to be *orthogonal* with respect to the weight function q on $[a, b]$ if

$$\int_a^b f(x)g(x)q(x)dx = 0$$

For a given function q, it is often possible to find an infinite sequence of functions $\phi_1, \phi_2, \ldots$ such that

$$\int_a^b \phi_n(x)\phi_m(x)q(x)dx = 0 \text{ if } m \neq n$$

The sequence $\{\phi_n\}_{n\geq 1}$ is called an *orthogonal system*. The norm of ϕ_n is defined as follows

$$\|\phi_n\| = \sqrt{\int_a^b \phi_n^2(x)q(x)dx}$$

Moreover if $\|\phi_n\| = 1$, $\{\phi_n\}_{n\geq 1}$ is called an *orthonormal system.*

Example 1.2.

1. $\{\sin(nx)\}_{n\geq 1}$ is an orthogonal system on the interval $[0, \pi]$ with respect to $q(x) = 1$. Indeed,

$$\int_0^\pi \sin(nx)\sin(mx)dx = \frac{1}{2}\int_0^\pi [\cos((m-n)x) - \cos((m+n)x)]\, dx$$

For $n \neq m$

$$= \left[\frac{1}{2(m-n)}\sin((m-n)x) - \frac{1}{2(m+n)}\sin((m+n)x)\right]_0^\pi = 0.$$

2. $\{\sqrt{\frac{2}{\pi}}\sin(nx)\}_{n\geq 1}$ is an orthonormal system.

2 Fourier Series

Let f be a piecewise continuous function defined on $[-\pi, \pi]$. Then one can express f as a linear combination of sine and cosine functions as follows :

$$f(x) \sim \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

where a_0, a_k and b_k are constants to be determined.

In order to find a_0, one has

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \left(\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \right) dx$$

$$= \frac{1}{2}a_0 \cdot 2\pi + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos(kx)dx + b_k \int_{-\pi}^{\pi} \sin(kx)dx \right)$$

$$\Longrightarrow \qquad a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx$$

On the other hand, to find a_n,

$$\int_{-\pi}^{\pi} f(x) \cos(nx)dx =$$

$$\int_{-\pi}^{\pi} \left(\frac{1}{2}a_0 \cos(nx) + \sum_{k=1}^{\infty} a_k \cos(kx) \cos(nx) + b_k \sin(kx) \cos(nx) \right) dx$$

$$= 0 + a_n \int_{-\pi}^{\pi} \cos(nx)\cos(nx)dx + 0 = a_n\pi$$

$$\implies a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx,\ n = 1, 2, \ldots$$

Similarly

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx,\ n = 1, 2, \ldots$$

a_n and b_n are called *Fourier coefficients.*

Example 2.1. Let us consider the function

$$f(x) = \begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

By using the formula of a_0, a_n and b_n, we find that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0}(-1)dx + \frac{1}{\pi}\int_{0}^{\pi}(1)dx = 0.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_{-\pi}^{0} -\cos(nx)dx + \frac{1}{\pi}\int_{0}^{\pi}\cos(nx)dx$$

$$= \frac{1}{\pi}\left([-\frac{1}{n}\sin(nx)]_{-\pi}^{0} + [\frac{1}{n}\sin(nx)]_0^{\pi}\right) = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_{-\pi}^{0} -\sin(nx)dx + \frac{1}{\pi}\int_{0}^{\pi}\sin(nx)dx$$

$$= \frac{1}{\pi}\left([\frac{1}{n}\cos(nx)]_{-\pi}^{0} + [-\frac{1}{n}\cos(nx)]_0^{\pi}\right)$$

$$= \frac{1}{n\pi}\left([\cos(0) - \cos(-n\pi)] + [\cos(0) - \cos(n\pi)]\right)$$

$$= \frac{2}{n\pi}(1 - \cos(n\pi)) = \frac{2}{n\pi}(1 - (-1)^n).$$

Thus the Fourier series of f is given by

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1 - (-1)^n\right)\sin(nx).$$

Theorem 2.1. If f is a periodic function with period 2π and f and f' are piecewise continuous on $[-\pi, \pi]$, then the fourier series

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

is convergent. The sum of the Fourier series is equal to $f(x)$ at all numbers x where f is continuous. At the numbers x where f is discontinuous, the sum of the Fourier series is the average of the right and the left limits, that is

$$\frac{1}{2}(f(x^+) + f(x^-)).$$

3 Fourier Sine + Cosine Series

In this section we show that the series of sines only (and the series of cosines only) are special cases of a Fourier series.

3.1 Fourier Sine Series

Definition 3.1. An *odd function* is a function with the property $f(-x) = -f(x)$.

Example 3.1.
1. $f(x) = x^3$.
2. $f(x) = \sin(4x)$.

Remark 3.1. The integral of an odd function over a symmetric interval is zero.

Let us calculate the Fourier coefficients of an odd function :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)dx$$

Since $a_n = 0$, all the cosine functions will not appear in the Fourier series of an odd function. The Fourier series of an odd function is an infinite series of odd functions (sines) :

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx)$$

Now assume that the function f is only given for $0 \leq x \leq \pi$ and not necessarily odd. In this case f can be extended as an odd function. This extension is called the *odd extension of f*. Moreover the Fourier series of the odd extension of f only involves sines :

$$\text{the odd extension of } f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx), \quad -\pi \leq x \leq \pi.$$

However, we are only interested in what happens $[0, \pi]$. In this interval f is identical to its odd extension :

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx), \quad 0 \leq x \leq \pi,$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

3.2 Fourier Cosine Series

Definition 3.2. An *even function* is a function with the property $f(-x) = f(x)$.

The sine coefficients of a Fourier series will be zero for an even function,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = 0.$$

The Fourier series of an even function is an infinite series of even functions (cosines) :

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx).$$

The coefficients of the cosines may be evaluated using information about $f(x)$ only on $[0, \pi]$, since

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx$$

If the function f is only given for $0 \leq x \leq \pi$ and not necessarily even, then f can be extended as an even function, which called the *even extension of* f. Moreover the Fourier series of the even extension of f only involves cosines :

$$\text{the even extension of } f(x) \sim \sum_{n=0}^{\infty} a_n \cos(nx), \quad -\pi \leq x \leq \pi.$$

In the interval $[0, \pi]$, the function f is identical to its even extension :

$$f(x) \sim \sum_{n=1}^{\infty} a_n \cos(nx), \quad 0 \leq x \leq \pi,$$

where

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

Example 3.2. Let us consider the function $f(x) = 1$ on $[0, \pi]$. The Fourier cosine series has coefficients

$$a_0 = \frac{2}{\pi} \int_0^{\pi} 1 dx = 2$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \cos(nx) dx = 0$$

Then $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = 1$.

4 Differentiation and Integration of Fourier Series

Let us consider a continuous function on the interval $[-\pi, \pi]$, with Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

then f' has a Fourier series

$$f'(x) \sim \sum_{n=1}^{\infty} -na_n \sin(nx) + nb_n \cos(nx).$$

That is $f'(x)$ can be written as a Fourier series

$$f'(x) \sim \sum_{n=1}^{\infty} \tilde{a}_n \cos(nx) + \tilde{b}_n \sin(nx),$$

where

$$\tilde{a}_n = b_n n = \frac{n}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$\tilde{b}_n = -a_n n = -\frac{n}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx.$$

For any function f on the interval $[-\pi, \pi]$, its integral has Fourier series (assume that $a_0 = 0$)

$$F(x) = \int^{x} f(\tilde{x}) d\tilde{x} \sim \frac{\tilde{a}_0}{2} + \sum_{n=1}^{\infty} \tilde{a}_n \cos(nx) + \tilde{b}_n \sin(nx)$$

where

$$\tilde{a}_n = -\frac{1}{n} b_n, \quad \tilde{b}_n = \frac{1}{n} a_n, \text{ and } \tilde{a}_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx.$$

Example 4.1. Let us consider the function

$$f(x) = \begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

The Fourier series is

$$\sum_{kodd} \frac{4}{k\pi} \sin(kx).$$

The Fourier series of

$$F(x) = \int_0^x f(\tilde{x}) d\tilde{x} = \begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 \le x \le \pi \end{cases},$$

is given by

$$F(x) = \frac{\pi}{2} + \sum_{kodd} \frac{-4}{k^2\pi} \cos(kx).$$

5 Complex Fourier Series

Recall that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ then

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Thus the Fourier series can be written using complex variables

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \left[\frac{e^{ikx} + e^{-ikx}}{2}\right] + b_k \left[\frac{e^{ikx} - e^{-ikx}}{2i}\right]$$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{(a_k - ib_k)}{2} e^{ikx} + \frac{(a_k + ib_k)}{2} e^{-ikx}$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{ikx} + c_{-k} e^{-ikx}$$

where

$$c_0 = \frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$c_k = \frac{(a_k - ib_k)}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)(\cos(kx) - i \sin(kx)) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i\pi x} dx$$

$$c_{-k} = \frac{(a_k + ib_k)}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)(\cos(kx) + i \sin(kx)) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{i\pi x} dx$$

In summary, the complex Fourier series is

$$f(x) \sim \sum_{-\infty}^{\infty} c_k e^{ikx}$$

where

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx.$$

Example 5.1. Let us consider the function

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi. \end{cases}$$

The complex Fourier series coefficients are

$$c_k = \frac{1}{2\pi}\left(\int_{-\pi}^{0} -1e^{-ikx}dx + \int_{0}^{\pi} 1e^{-ikx}dx\right)$$

$$= \begin{cases} \frac{1}{2\pi}\left(\frac{-1}{-ik}e^{-ikx}]_{-\pi}^{0} + \frac{1}{-ik}e^{-ikx}]_{0}^{\pi}\right), & k \neq 0 \\ \frac{1}{2\pi}\left(-\pi + \pi\right), & k = 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2ki\pi}\left(1 - e^{ik\pi} - e^{-ik\pi} + 1\right), & k \neq 0 \\ 0, & k = 0 \end{cases}$$

$$= \begin{cases} \frac{-i}{k\pi}\left(1 - \frac{1}{2}(e^{ik\pi} - e^{-ik\pi})\right), & k \neq 0 \\ 0, & k = 0 \end{cases}$$

$$= \begin{cases} \frac{-i}{k\pi}\left(1 - \cos(k\pi))\right), & k \neq 0 \\ 0, & k = 0 \end{cases}$$

$$= \begin{cases} \frac{-2i}{k\pi}, & k \; odd \\ 0, & k \; even \end{cases}$$

Then

$$f(x) \sim \sum_{kodd} \frac{-2i}{k\pi} e^{ikx}.$$

6 Fourier Series over any Interval

In general, Fourier series, sine and cosine series may be defined not just over $[-\pi, \pi]$ but over any interval $[a, b]$. Let us consider a function $F(t)$ periodic with a period 2π and $t \in [-\pi, \pi]$, then the Fourier series of F is given by

$$F(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kt) + b_k \sin(kt),$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)dt \text{ and } b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)dt$$

Let a and b be two constants. Define the new variable x as follows

$$x = \frac{1}{2}(b+a) + \frac{1}{2\pi}(b-a)t \in [a,b]$$

$$t = \pi\left(\frac{2x-b-a}{b-a}\right).$$

Now let us define the function f by

$$f(x) = f\left(\frac{1}{2}(b+a) + \frac{1}{2\pi}(b-a)t\right) = F(t)$$

The Fourier series of the function f is

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(k\pi\frac{(2x-b-a)}{b-a}\right) + b_k \sin\left(k\pi\frac{(2x-b-a)}{b-a}\right)$$

where

$$a_k = \frac{2}{b-a}\int_a^b f(x)\cos\left(k\pi\frac{(2x-b-a)}{b-a}\right)dx$$

and

$$b_k = \frac{2}{b-a}\int_a^b f(x)\sin\left(k\pi\frac{(2x-b-a)}{b-a}\right)dx.$$

In particular, if $b = l$, $a = -l$ and f is periodic with period $2l$

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{l}\right) + b_k \sin\left(\frac{k\pi x}{l}\right)$$

where

$$a_k = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{k\pi x}{l}\right)dx \text{ and } b_k = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{k\pi x}{l}\right)dx$$

Example 6.1. Let us consider the function f defined as follows

$$f(x) = \begin{cases} 0, & -2 \le x < 0 \\ 2-x, & 0 < x \le 2. \end{cases}$$

By using the formula of a_0, a_n and b_n, we find that

$$a_0 = \frac{1}{4}\int_0^2 (2-x)dx,$$

$$a_k = \frac{1}{2}\int_0^2 (2-x)\cos\frac{k\pi x}{2}dx, k = 1, 2, \ldots$$

and

$$b_k = \frac{1}{2}\int_0^2 (2-x)\sin\frac{k\pi x}{2}dx, k = 1, 2, \ldots$$

Evaluating these integrals gives

$$a_0 = \frac{1}{2},\ a_k = \frac{2}{k^2\pi^2}[1-(-1)^k] \text{ and } b_k = \frac{2}{k\pi}$$

where use has been made of the fact that $\cos(k\pi) = (-1)^k$ and $\sin(k\pi) = 0$. Thus the Fourier series becomes

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\left(\frac{[1-(-1)^k]}{k^2\pi}\cos\frac{k\pi x}{2} + \frac{1}{k}\sin\frac{k\pi x}{2}\right).$$

The sine series of a function f on $[0, l]$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi x}{l}$$

where

$$a_k = \frac{2}{l} \int_0^l f(x) \cos \frac{k\pi x}{l} dx.$$

The cosine series of a function f on $[0, l]$ is

$$f(x) \sim \sum_{k=1}^{\infty} a_k \sin \frac{k\pi x}{l}$$

where

$$a_k = \frac{2}{l} \int_0^l f(x) \sin \frac{k\pi x}{l} dx.$$